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On the Satellite Line of the Cubic.

By R. M. Winger.

The Rational Cubic.

1. *Introduction.* While Salmon * discusses the satellite line of the cubic at some length he fails to give its equation. The equation was supplied by Cayley † who calculated it laboriously by a direct method. Several covariant expressions for this remarkable line were given by Walker.‡ And another has been furnished by Morley.§ In the present paper the explicit equation of the satellite is exhibited both for the rational and the general cubic, in canonical form, and some associated loci are considered. Several interesting chain theorems are obtained and a generalization is made for the plane curve of order n.

We consider first the rational cubic R_4^3, taking as its equations in points and lines respectively

$$x_0 = 3t^2, \quad x_1 = 3t, \quad x_2 = t^3 + 1, \tag{1}$$

or

$$x_0^3 + x_1^3 - 3x_0x_1x_2 = 0,$$

$$u_0 = 1 - 2t^3, \quad u_1 = t^4 - 2t, \quad u_2 = 3t^2, \tag{2}$$

or

$$u_2^4 - 6u_0u_1u_2^2 - 3u_0^2u_1^2 + 4u_0^3u_2 + 4u_1^3u_2 = 0,$$

when the triangle of reference is the invariant triangle of the G_6 which leaves the curve unaltered.

An important curve for the satellite theory is the conic N, || the locus of lines joining pairs of contacts of tangents from points of the R_4^3. The equation of N for the cubic (1) is

$$u_2^2 - 9u_0u_1 = 0. \tag{3}$$

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* *Higher Plane Curves*, third edition, Art. 149 ff.

† "A Memoir on Curves of the Third Order," *Phil. Trans.* (1857), p. 439, or *Collected Papers*, Vol. II, p. 405.

‡ *Phil. Trans.* (1888) A, p. 170 and *Proc. Lond. Math. Soc.*, Vol. XXI (1890), p. 247.

§ University lectures for the session 1910-1911.

|| Morley, American Journal of Mathematics, Vol. XI (1889), p. 316; Winger, "Involutions on the Rational Cubic," *Bulletin, Amer. Math. Soc.*, October, 1918, p. 28. This conic touches the nodal tangents where they meet the line of flexes and has contacts with the curve at the sextactic points.

While every line u' has a unique satellite u, a line u is in general satellite to four lines u'. Continuing the figure we shall call the line u' the *primary* of u. Combining these statements we have the theorem: *The* 6 *contacts of tangents from* 3 *collinear points* (*on* u) *of a rational cubic lie by threes on the* 4 *sides* u' *of a quadrilateral whose diagonal* 3-*line is circumscribed to* N.

When u' is tangent to the curve, the satellite is likewise tangent, at the point where u' cuts again. If however u is tangent two of the primary lines coincide. Thus the tangent at a point is satellite to the two tangents from the point and the line, counted twice, joining contacts of those tangents.

2. The line $u' \equiv (ux) \equiv u_0x_0 + u_1x_1 + u_2x_2 = 0$

cuts the curve in three points with parameters

$$u_2t^3 + 3u_0t^2 + 3u_1t + u_2 = 0. \tag{4}$$

If s_i refer to the symmetric functions of these t's and σ_i to the parameters $(-1/t^2)$ of the three tangential points we have

$$\begin{gathered} s_1 = -3u_0/u_2, \quad s_2 = 3u_1/u_2, \quad s_3 = -1, \\ \sigma = -\Sigma\, 1/t_i^2 = -(2s_1 + s_2^2), \quad \sigma_2 = s_1^2 - 2s_2, \quad \sigma_3 = -1. \end{gathered} \tag{5}$$

Whence the equation giving the tangential points is

$$t^3 + (2s_1 + s_2^2)t^2 + (s_1^2 - 2s_2)t + 1 = 0 \tag{6}$$

and the equation of the line u on which they lie is

$$\Sigma_2\colon\ (3u_1^2 - 2u_0u_2)x_0 + (3u_0^2 - 2u_1u_2)x_1 + u_2^2x_2 = 0. \tag{7}$$

For given u' (7) *is the equation of the satellite line. Equally for given* x *and variable* u *it is the equation of a conic, the locus of lines whose satellites pass through* x. When considered as a conic we shall denote (7) by Σ_2.

If x is on the curve the conic degenerates to the two contacts of tangents from x. Hence the discriminant of the conic is to within a factor the point equation of the R_4^3.

3. If x is not on the cubic 4 tangents u_i can be drawn to the curve. From each point of contact t_i canbe drawn 2 tangents u'_{i1} and u'_{i2}, with contacts t'_{i1} and t'_{i2}. These tangents u' and the line u'_{in} joining their contacts are all primaries of u_i. *Hence the four pairs of tangents* u'_{i1}, u'_{i2} *which can be drawn to the* R_4^3 *from the contacts* t_i *of tangents from a point* x, *together with the four lines* u'_{in} *joining contacts* t'_{i1}, t'_{i2} *of the pairs touch a conic* Σ_2. *The four pairs of tangents are the common lines of conic and* R_4^3 *and the four lines* u'_{in} *are the common lines of* Σ_2 *and conic* N.

Continuing, from each of the 8 points t' can be drawn a pair of tangents u'' with contacts t'' and these contacts determine a line u''_n. *These* 24 *lines all touch a curve of class four, viz. the locus of lines whose satellites envelop the conic* Σ_2. *The* 16 *lines* u'' *are the common lines of the quartic and the* R_4^3 *while the* 8 *lines* u''_n *are the common lines of the quartic and the conic* N.

Starting with a curve Σ_1 of class one, namely the point x, we have thus a chain of theorems on class curves associated with the rational cubic for the process can be continued indefinitely. This chain is however a special case of another which can be obtained by direct generalization of the theorem implied by equation (7). Since the equation of the satellite line is of the second degree in the coefficients of the primary, we may say: *If the satellite line* u *run around a curve* Σ_δ *of class* δ, *the primary* u' *envelops a curve* $\Sigma_{2\delta}$ *of class* 2δ. *The equation of* $\Sigma_{2\delta}$ *can be found at once by replacing* u_0, u_1, u_2 *in the homogeneous line equation of* Σ_δ *by*

$$(3u_1^2-2u_0u_2) : (3u_0^2-2u_1u_2) : u_2^2$$

respectively.

Now the R_4^3 and Σ_δ have 4δ common lines u. If the contacts be designated by t, then from each t can be drawn two tangents u' to the cubic. The contacts t' of these tangents determine a line u'_n. These lines u' are the three primaries of u and hence are lines of $\Sigma_{2\delta}$. *We have thus* 12δ lines of $\Sigma_{2\delta}$,—*the* 8δ *lines* u' *which are the common lines of* $\Sigma_{2\delta}$ *and* R_4^3 *and the* 4δ *lines* u'_n *which the common lines of* $\Sigma_{2\delta}$ *and the conic* N. From the 8δ points t' can be drawn in turn 8δ pairs of tangents u'' which determine 8δ lines u''_n joining contacts t'' of the pairs. *These lines all touch a curve* $\Sigma_{4\delta}$ *of class* 4δ, *the locus of lines whose satellites envelop* $\Sigma_{2\delta}$. *The* 16δ *lines* u'' *are the common lines of* $\Sigma_{4\delta}$ *and the cubic and the* 8δ *lines* u''_n *are the common lines of* $\Sigma_{4\delta}$ *and the conic* N. *And so on ad infinitum.* Hence any curve Σ_δ establishes a chain which reduces to the old when $\delta = 1$.

4. We come now to the converse problem: given a class curve to find the satellite locus. Theorem. *If a line touch a curve* S_δ *of class* δ, *the satellite envelops a curve* $S_{2\delta}$ *of class* 2δ.* *In particular if* S_δ *is rational so also is* $S_{2\delta}$. If we denote the satellite locus by S_μ, then we want the number of lines of S_μ which pass through an arbitrary point. Now those lines u' of S_δ whose satellites pass through a given x touch a conic Σ_2. But S_δ and the conic have just 2δ lines in common i. e. there are 2δ lines of S_μ on x and $\mu = 2\delta$.

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* This involves no contradiction with the theorem of the previous section; for each line u has four primaries u', hence as u' runs around $\Sigma_{2\delta}$, u will generate an $S_{4\delta}$, which is Σ_δ repeated four times.

To prove the second part of the theorem, let

$$u_0(t)x_0 + u_1(t)x_1 + u_2(t)x_2 = 0, \tag{8}$$

where the u's are binary forms of order δ be the map equation in lines of a rational curve R_δ of class δ. The map equation of the satellite locus is found by replacing the coefficients of the x's by

$$(3u_1^2 - 2u_0u_2) : (3u_0^2 - 2u_1u_2) : u_2^2$$

respectively. The result is obviously an equation in which the coefficients of the x's are rational and of degree 2δ in t.

To find the common lines of $S_{2\delta}$ and R_4^3 it is only necessary to recall that the satellite is tangent to the cubic only if the primary is a tangent of the cubic or a line of conic N. S_δ and R_4^3 have 4δ common lines whose satellites, viz. the tangents at the points in which the lines cut R_4^3 again, account for 4δ common lines of R_4^3 and $S_{2\delta}$. The 2δ primaries which touch the conic N therefore must take up the remaining 4δ common lines, i. e. each of these satellites will count for two.

As an application of these theorems we observe

(*a*) *The satellite locus of the conic N is the cubic itself.*

(*b*) *The cubic taken twice is its own satellite locus.*

(*c*) *The locus of lines whose satellites touch the cubic is a composite curve of class eight consisting of the cubic and the conic N repeated.*

THE NON-SINGULAR CUBIC.

5. Most of the theorems stated for the rational cubic can be extended at once to the case of the general cubic C_6^3. The tangent at a point P is now satellite to 10 lines, viz. the 4 tangents from the point and the 6 lines (repeated) joining in pairs the contacts of these tangents. Salmon * has remarked that the envelope of the 6 lines for variable P is a certain composite curve M_9 consisting of three class cubics.

Morley † has shown that the satellite of the line (ux) is

$$\{\Omega^2 - \tfrac{1}{6}(ux)\Omega^3\}\, C^3\Gamma_6 \tag{9}$$

where C^3 and Γ_6 are respectively the point and line equations of the curve and Ω is the ternary differential operator

$$\left(\frac{\partial^2}{\partial x_0 \partial u_0} + \frac{\partial^2}{\partial x_1 \partial u_1} + \frac{\partial^2}{\partial x_2 \partial u_2}\right)$$

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* *Higher Plane Curves*, Art. 151.

† Loc. cit.

Calculated for the canonical form

$$x_0^3 + x_1^3 + x_2^3 + 6ax_0x_1x_2 = 0, \tag{10}$$

the equation of the satellite line is

$$\Gamma_4 : \sum_0^2 \{u_i^4 - 2(u_j^3 + u_k^3)u_i - 6a\, u_j^2 u_k^2\} x_i = 0, \quad i \neq j \neq k \tag{11}$$

Again for given x equation (11) represents a class quartic Γ_4, the locus of lines whose satellites are on x. This curve is of the Humbert or *desmic* type.* A line quartic is desmic if it belongs to a pencil of quartics which contains three degenerate curves each composed of four points, the three degenerate curves then being desmic quadrangles. That Γ_4 is desmic can be shown as follows. Consider a line u cutting the cubic in points a, b and c. Denote the contacts of tangents from these points by α_i, β_i, γ_i, $(i = 1, 2, 3, 4)$. Any point x on u will determine a curve Γ_4. Two points x and x' thus determine a pencil

$$\Gamma_4 + \lambda \Gamma'_4 \tag{12}$$

of curves associated with the points $x + \lambda x'$ of u. Now the primary lines of the (line) pencil on a are obviously the four (line) pencils on points α_i. In other words points α constitute a degenerate member of the pencil (12). This pencil of quartics thus contains the three desmic quadrangles α, β, γ which proves the theorem.

We have the following theorems for the general cubic, omitting the proofs which follow closely those given at length for the earlier case.

If the satellite line run around a curve Γ_δ of class δ, the primary envelops a curve $\Gamma_{4\delta}$ of class 4δ whose equation is obtained by replacing u_0, u_1, u_2 in the homogenous equation of Γ_δ by the coefficients of the x's in equation (11). Γ_δ and C_6^3 have 6δ common lines u with contacts s. From each point s can be drawn to the cubic 4 tangents u' with contacts s'. Joining the points s' in pairs are 6 lines u'_m. There are thus 60δ tangents of $\Gamma_{4\delta}$ which comprise the primaries of lines u,—the 24δ tangents u' which are the common lines of $\Gamma_{4\delta}$ and the cubic, and the 36δ lines u'_m which are the common lines of $\Gamma_{4\delta}$ and the curve M_9.

This is the first link in a chain of theorems associated with every curve Γ_δ. A chain of especial interest is that originating with a point Γ_1:

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* After Humbert who discusses such curves in two papers, *Journal de Mathématiques*, 4e Série, Vol. VI (1890), p. 423 and VII (1891), p. 353. Professor Morley called my attention to the fact that Γ_4 is desmic. Since it depends on eleven constants it would appear to be the general desmic quartic.

From a point x can be drawn in general 6 tangents u with contacts s. From each point s can be drawn 4 tangents u' with contacts s' and the points s' can be joined in pairs by 6 lines u'_m. The 24 tangents u' and 36 lines u'_m touch a curve Γ_4 of class four, namely the locus of lines whose satellites pass through x. The tangents u' are the common lines of Γ_4 and C_6^3 while the lines u'_m are the common lines of Γ_4 and M_9. Again the 24 points s' determine 96 tangents u'' with contacts s'' and 144 lines u''_m joining these contacts in pairs. These 240 lines u'' touch a curve Γ_{16}, of class 16, the primary locus of Γ_4, and comprise the lines which Γ_{16} has in common with C_6^3 and M_9. And so on forever.

Most of the peculiarities of Humbert's curve as summarized (for the dual) in section 12 of his first paper can be recovered readily with our present apparatus. Indeed it seems preferable to reverse his procedure and derive the properties of the quartic from the cubic and point which define it.

The six characteristic 4-points whose diagonal points are nodes and whose connecting lines are the nodal tangents are the six sets of points s' in the theorem just stated. In other words, *the 36 intersections of Γ_4 and the cubic are at these 18 nodes and the common lines of Γ_4 and M_9 are the 36 nodal tangents,—each component of M_9 touching the tangent pair of one node in each set.* Or again *from the tangentials of the points s can be drawn 18 tangents (in addition to the lines u), the contacts of which are at the 18 nodes.*

Likewise the satellite locus of a curve C_δ of class δ is a curve $C_{4\delta}$ of class 4δ rational if C_δ is rational. The common lines of $C_{4\delta}$ and C_6^3 are (a) the satellites of the 6δ common lines of C_δ and the cubic and (b) the satellites of the 9δ common lines of C_δ and M_9, the latter lines each counting for two.

The satellite locus of M_9 is the cubic repeated six times.

The satellite locus of the cubic is the curve itself four times repeated; while the primary locus consists of the cubic and M_9 repeated.

Since M_9 is the double curve in the primary locus of the cubic it must be the Jacobian of the coefficients in the satellite line.* Calculated thus and verified for special lines the equation of M_9 is found to be

$$U^3 - 36a^2U^2 - 54aU - 54(1 + 4a^3) = 0 \tag{13}$$

where $$U = \frac{u_0^3 + u_1^3 + u_2^3}{u_0u_1u_2}.$$

Hence the factors of M_9 are

$$u_0^3 + u_1^3 + u_2^3 - k_iu_0u_1u_2 = 0, \tag{14}$$

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* Likewise the Jacobian of the coefficients in the satellite (7) of the rational cubic is Nu_2 which incidentally proves the defining property of N.

where k_i are the roots of (13) considered as a cubic in U. It follows that the three components belong to the syzygetic pencil of class cubics determined by the nine harmonic polars.

M_9 also touches the cubic at the 27 sextactic points. Hence the tangents to the cubic from the flexes account for all the common lines of the two curves.

THE GENERAL PLANE CURVE.

6. Consider now the plane curve C_δ^n of order n and class δ. The tangents at n points of a line (ux) constitute a curve T^n which meets C^n in n^2 points. Since $2n$ of these points are the complete intersections of C^n and a curve of order 2,—the line repeated,—it follows from the theory of residuation that the remaining $n(n-2)$ lie on a C^{n-2}, the satellite $(n-2)$-ic of the line u.

The equation of the satellite curve, $f(u^k x^{n-2})=0$, will contain u as well as x. For a given x therefore, $f=0$ represents a curve of class k, the locus of lines whose satellites pass through x. To ascertain the value of k, it will be sufficient to enumerate the common lines of C_μ^n and f considered as a class curve. Let u' be such a line. Then since u' is a tangent to C^n its satellite curve degenerates into the $n-2$ tangents at the remaining intersections of u' and C^n. One of these $n-2$ lines must pass through x in virtue of the defining property of f, i. e. u' is a tangent to C^n from a contact of one of the tangents from x. Since C_δ^n is of class δ, there are precisely $\delta(\delta-2)$ lines u'. Hence $k=\delta-2$.*

We have at once the following generalization of the first link of the special chain theorem for the cubic. *Denoting by* $t_1, t_2, \cdots\cdots t_\delta$ *the contacts of the* δ *tangents from an arbitrary point* x *to a curve* C_δ^n *of order* n *and class* δ, *the* δ *sets of* $(\delta-2)$ *tangents from points* t_i *comprise the common lines of* C_δ^n *and a curve* f *of class* $(\delta-2)$, *namely the locus of lines whose satellite* $(n-2)$*-ics pass through* x. If C_δ^n is non-singular f is of class $(n+1)(n-2)$ and there are $(n+1)(n)(n-1)(n-2)$ common lines.

UNIVERSITY OF OREGON,
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* It can be shown by the analytic method employed by Morley, l. c. that f involves the coefficients of C_δ^n to the degree $2n-1$. For an application of that method when $n=4$, see a paper on the satellite conic of the quartic by T. Cohen, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVIII (1916), p. 325.